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# New sharp bounds for logarithmic mean and identric mean

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**Abstract**

For $x, y > 0$ with $x \neq y$, let $L = L(x, y)$, $I = I(x, y)$, $A = A(x, y)$, $G = G(x, y)$, $A_r = A^{1/r}(x^r, y^r)$ denote the logarithmic mean, identric mean, arithmetic mean, geometric mean and $r$-order power mean, respectively. We find the best constant $p, q > 0$ such that the inequalities

$$A_p^{1/(3p)} G^{1-1/(3p)} < L < A_q^{1/(3q)} G^{1-1/(3q)},$$
$$A_p^{2/(3p)} G^{1-2/(3p)} < I < A_q^{2/(3q)} G^{1-2/(3q)}$$

hold, respectively. From them some new inequalities for means are derived. Lastly, our new lower bound for the logarithmic mean is compared with several known ones, which shows that our results are superior to others.

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**Keywords:** logarithmic mean; identric mean; power mean; inequality

## 1 Introduction

The logarithmic and identric means of two positive real numbers $x$ and $y$ with $x \neq y$ are defined by

$$L = L(x, y) = \frac{x - y}{\ln x - \ln y} \quad \text{and} \quad I = I(x, y) = e^{-1} \left( \frac{x^x}{y^y} \right)^{1/(x-y)},$$

respectively. The power mean of order $r$ of the positive real numbers $x$ and $y$ is defined by

$$A_r = A_r(x, y) = \left( \frac{x^r + y^r}{2} \right)^{1/r} \quad \text{if } r \neq 0 \text{ and } A_0 = A_0(x, y) = \sqrt{xy}.$$

The main properties of these means are given in [1]. In particular, the function $r \mapsto A_r(x, y)$ $(x \neq y)$ is continuous and strictly increasing on $\mathbb{R}$. As special cases, the arithmetic mean and geometric mean are $A = A(x, y) = A_1(x, y)$ and $G = G(x, y) = A_0(x, y)$, respectively.

In the recent past, the logarithmic mean and the identric mean have been the subject of intensive research. Ostle and Terwilliger [2] and Karamata [3] first proved that

$$G < L < A. \tag{1.1}$$

This result, or a part of it, has been rediscovered and reproved many times (see, *e.g.*, [4–7]).

In 1974 Lin [8] obtained an important refinement of the above inequalities:

$$G < L < A_{1/3}, \tag{1.2}$$

and proved that the number 1/3 cannot be replaced by a smaller one. A sharpness of the second inequality in (1.2) has been shown by Neuman [9]. The following inequality is due to Carlson [10]:

$$L > \sqrt{A_{1/2}G}. \tag{1.3}$$

In [11], the authors present a very nice double inequality, that is,

$$A^{1/3}G^{2/3} < L < \frac{1}{3}A + \frac{2}{3}G. \tag{1.4}$$

Using a new method, Wang and Wang [12] proved that

$$L > A_p^{1-p}G^p \tag{1.5}$$

holds for $p = 0, 1/2, 1/3$. Chen and Wang [13] pointed out this inequality is true for all real numbers $p$. Only when $p \in (0,1)$, however, the inequality $A_p^{1-p}G^p > G$ would be true. In 2009, another better lower bound for $L$ was given by Zhu [14], that is,

$$L > \left(\frac{7}{15}A + \frac{8}{15}G\right)^{5/7} G^{2/7} > A_{1/2}^{2/3}G^{1/3} > \left(\frac{2}{3}A + \frac{1}{3}G\right)^{1/2} G^{1/2} > A^{1/3}G^{2/3}. \tag{1.6}$$

The following lower bound for $L$ in terms of $I$ and $G$ is due to Alzer [15]:

$$L > \sqrt{IG}. \tag{1.7}$$

Very recently, Yang [16] showed that

$$L > A_{1/2}^{2/3}G^{1/3} > \sqrt{IG} > A^{1/3}G^{2/3}. \tag{1.8}$$

For the identric mean $I$, Stolarsky [17, 18] and Pittenger [19] presented lower and upper bounds for $I$ as follows:

$$G < L < I < A, \tag{1.9}$$

$$A_{2/3} < I < A_{\ln 2}, \tag{1.10}$$

and the constants 2/3 and $\ln 2$ are the best possible. Inequalities (1.9) were also rediscovered by Yang [6]. The following result is due to Sándor [20]:

$$I > \frac{2A + G}{3} > A^{2/3}G^{1/3}. \tag{1.11}$$

Other inequalities for $L$ and $I$ and their applications can be found in the literature [20–33].

The aim of this paper is to find the best $p, q \in \mathbb{R}$ such that the inequalities

$$A_p^{1/(3p)} G^{1-1/(3p)} < L < A_q^{1/(3q)} G^{1-1/(3q)}, \tag{1.12}$$

$$A_p^{2/(3p)} G^{1-2/(3p)} < I < A_q^{2/(3q)} G^{1-2/(3q)} \tag{1.13}$$

hold.

It is easy to check that both the functions

$$p \mapsto A_p^{1/(3p)} G^{1-1/(3p)}, \tag{1.14}$$

$$p \mapsto A_p^{2/(3p)} G^{1-2/(3p)} \tag{1.15}$$

are even on $(-\infty, \infty)$, and therefore we assume that $p, q > 0$ in what follows.

Our main results are stated as follows.

**Theorem 1** *Let $p, q > 0$. Then inequalities (1.12) hold for all $x, y > 0$ with $x \neq y$ if and only if $p \geq p_0 = 1/\sqrt{5}$ and $0 < q \leq 1/3$, and the function $p \mapsto A_p^{1/(3p)} G^{1-1/(3p)}$ is decreasing on $(0, \infty)$.*

**Theorem 2** *Let $p, q > 0$. Then inequalities (1.13) hold for all $x, y > 0$ with $x \neq y$ if and only if $p \geq 2/3$ and $0 < q \leq q_0 = \sqrt{10}/5 = 0.63246$, and the function $p \mapsto A_p^{2/(3p)} G^{1-2/(3p)}$ is decreasing on $(0, \infty)$.*

We will prove two theorems above by hyperbolic function theory. For this end, we need the following lemma, which tells us an inequality for bivariate homogeneous means can be equivalently changed into the form of hyperbolic functions.

**Lemma 1** *Let $M(x, y)$ be a homogeneous mean of positive arguments $x$ and $y$. Then*

$$M(x, y) = \sqrt{xy} M\left(e^t, e^{-t}\right),$$

*where $t = \frac{1}{2} \ln(x/y)$.*

By symmetry, we assume that $x > y > 0$. Then we have

$$L\left(e^t, e^{-t}\right) = \frac{\sinh t}{t}, \qquad I\left(e^t, e^{-t}\right) = e^{\frac{t \cosh t}{\sinh t} - 1},$$

$$A_p\left(e^t, e^{-t}\right) = \cosh^{1/p} pt, \qquad G\left(e^t, e^{-t}\right) = 1,$$

where $t = \frac{1}{2} \ln(x/y) > 0$. And then, due to Lemma 1, Theorem 1 and Theorem 2 can be restated as equivalent ones, respectively.

**Theorem 1′** *Let $p, t > 0$. Then the inequality*

$$\frac{\sinh t}{t} > (\cosh pt)^{1/(3p^2)} \tag{1.16}$$

*holds for all* $t > 0$ *if and only if* $p \geq p_0 = 1/\sqrt{5}$ *and the function* $p \mapsto (\cosh pt)^{1/(3p^2)}$ *is decreasing on* $(0, \infty)$. *Inequality* (1.16) *is reversed if and only if* $0 < p \leq 1/3$.

**Theorem 2′** *Let* $p, t > 0$. *Then the inequality*

$$e^{\frac{t\cosh t}{\sinh t}-1} > (\cosh pt)^{2/(3p^2)} \tag{1.17}$$

*holds for all* $t > 0$ *if and only if* $p \geq 2/3$ *and the function* $p \mapsto (\cosh pt)^{2/(3p^2)}$ *is decreasing on* $(0, \infty)$. *Inequality* (1.17) *is reversed if and only if* $0 < p \leq q_0 = \sqrt{10}/5$.

Therefore, we will prove Theorem 1′ and Theorem 2′ instead of Theorem 1 and Theorem 2 in the sequel.

## 2 Proof of Theorem 1′

In order to prove Theorem 1′, we first give the following lemmas.

**Lemma 2** *For* $t > 0$, *let the function* $U : (0, \infty) \mapsto (0, \infty)$ *be defined by*

$$U(p) = p^{-2} \ln \cosh pt. \tag{2.1}$$

*Then* $U$ *is decreasing on* $(0, \infty)$ *with*

$$\lim_{p \to 0^+} U(p) = \frac{1}{2} t^2, \qquad \lim_{p \to \infty} U(p) = 0.$$

*Proof* Differentiation yields

$$p^3 U'(p) = -2 \ln(\cosh pt) + pt \frac{\sinh pt}{\cosh pt} := V(p),$$

$$V'(p) = -\frac{1}{2} \frac{t}{\cosh^2 pt} (\sinh 2pt - 2pt) < 0,$$

which implies that $V$ is decreasing on $(0, \infty)$, and so $V(p) < \lim_{p \to 0} V(p) = 0$. Therefore, $U'(p) < 0$, that is to say, $U$ is decreasing on $(0, \infty)$. And, by L'Hospital's rule, we have

$$\lim_{p \to 0} U(p) = \lim_{p \to 0} \frac{1}{2p} \frac{t \sin pt}{(\cosh pt)} = \frac{1}{2} t^2, \qquad \lim_{p \to \infty} U(p) = \lim_{p \to \infty} \frac{1}{2p} \frac{t \sin pt}{(\cosh pt)} = 0,$$

which proves the lemma. □

**Remark 1** From Lemmas 1 and 2 it follows that the function $p \mapsto (A_p/G)^{1/p}$ is decreasing on $(0, \infty)$, and

$$\lim_{p \to 0} (A_p/G)^{1/p} = G \exp\left( \frac{1}{8} \ln^2(x/y) \right), \qquad \lim_{p \to \infty} (A_p/G)^{1/p} = 1.$$

Since

$$A_p^{1/(3p)} G^{1-1/(3p)} = \left( (A_p/G)^{1/p} \right)^{1/3} G, \qquad A_p^{2/(3p)} G^{1-2/(3p)} = \left( (A_p/G)^{1/p} \right)^{2/3} G,$$

so are the functions defined by (1.14) and (1.15), and

$$\lim_{p\to 0} A_p^{1/(3p)} G^{1-1/(3p)} = Ge^{(\ln(x/y))^2/24}, \qquad \lim_{p\to\infty} A_p^{1/(3p)} G^{1-1/(3p)} = G, \tag{2.2}$$

$$\lim_{p\to 0} A_p^{2/(3p)} G^{1-2/(3p)} = Ge^{(\ln(x/y))^2/12}, \qquad \lim_{p\to\infty} A_p^{1/(3p)} G^{1-1/(3p)} = G. \tag{2.3}$$

**Lemma 3** *Let $p > 0$ and let $f : (0,\infty) \mapsto (-\infty,\infty)$ be the function defined by*

$$f(t) = \ln \frac{\sinh t}{t} - \frac{1}{3p^2} \ln \cosh pt. \tag{2.4}$$

*Then we have*

$$\lim_{t\to 0^+} \frac{f(t)}{t^4} = \frac{1}{36}\left(p^2 - \frac{1}{5}\right), \tag{2.5}$$

$$\lim_{t\to\infty} \frac{f(t)}{t} = \frac{1}{p}\left(p - \frac{1}{3}\right). \tag{2.6}$$

*Proof* Using L'Hospital's rule gives (2.5). To obtain (2.6), we write $f(t)$ as

$$f(t) = t\left(1 - \frac{1}{3p} - \frac{\ln t}{t}\right) + \ln \frac{1 - e^{-2t}}{2} - \frac{1}{3p^2} \ln \frac{1 + e^{-2pt}}{2},$$

from which (2.6) easily follows.

This lemma is proved. □

**Lemma 4** *For $p > 0$, let $f$ be defined by* (2.4). *Then $f$ is increasing if $p \ge 1/\sqrt{5}$ and decreasing if $0 < p \le 1/3$.*

*Proof* Differentiation yields

$$\begin{aligned} f'(t) &= \frac{t}{\sinh t}\left(\frac{1}{t}\cosh t - \frac{1}{t^2}\sinh t\right) - \frac{1}{3p\cosh pt}\sinh pt \\ &= \frac{-3p\cosh pt \sinh t - t\sinh pt \sinh t + 3pt\cosh pt \cosh t}{3pt\cosh pt \sinh t} \\ &:= \frac{f_1(t)}{3pt\cosh pt \sinh t}, \end{aligned} \tag{2.7}$$

where

$$f_1(t) = -3p\cosh pt \sinh t - t\sinh pt \sinh t + 3pt\cosh pt \cosh t.$$

Using 'product into sum' formula for hyperbolic functions gives

$$\begin{aligned} f_1(t) = {} & \left(\frac{1}{2} + \frac{3}{2}p\right) t\cosh t(p-1) + \frac{3}{2}p\sinh t(p-1) \\ & + \left(\frac{3}{2}p - \frac{1}{2}\right) t\cosh t(p+1) - \frac{3}{2}p\sinh t(p+1), \end{aligned}$$

and expanding $f_1(t)$ in power series gives

$$\begin{aligned} f_1(t) &= \left(\frac{1}{2}+\frac{3}{2}p\right)\sum_1^\infty \frac{(p-1)^{2n-2}t^{2n-1}}{(2n-2)!} + \frac{3}{2}p\sum_1^\infty \frac{(p-1)^{2n-1}t^{2n-1}}{(2n-1)!} \\ &\quad + \left(\frac{3}{2}p-\frac{1}{2}\right)\sum_1^\infty \frac{(p+1)^{2n-2}t^{2n-1}}{(2n-2)!} - \frac{3}{2}p\sum_1^\infty \frac{t^{2n-1}(p+1)^{2n-1}}{(2n-1)!} \\ &:= \sum_1^\infty \frac{u(p,n)a_n}{2(2n-1)!}(p+1)^{2n-2}t^{2n-1}, \end{aligned} \tag{2.8}$$

where

$$a_n = \left(\frac{p-1}{p+1}\right)^{2n-2} - \frac{u(-p,n)}{u(p,n)},$$
$$u(p,n) = 2(3p+1)n + \left(3p^2-6p-1\right).$$

It is easy to check that

$$u(p,n) = (6p+2)\left(n-1+\frac{3p^2+1}{6p+2}\right) > 0 \tag{2.9}$$

and

$$a_{n+1} = \left(\frac{p-1}{p+1}\right)^2 a_n + v(n,p), \tag{2.10}$$

where

$$\begin{aligned} v(n,p) &= \left(\frac{p-1}{p+1}\right)^2 \frac{u(-p,n)}{u(p,n)} - \frac{u(-p,n+1)}{u(p,n+1)} \\ &:= \frac{16p(n-1)}{(p+1)^2u(p,n)u(p,n+1)}w(n,p), \end{aligned} \tag{2.11}$$

here

$$w(n,p) = 3(3n-1)p^2 - (n+1). \tag{2.12}$$

Now we are ready to prove desired results.

(i) We first prove that $f$ is increasing if $p \ge 1/\sqrt{5}$. To this end, by (2.7) in combination with (2.8) and (2.9), it suffices to show that $a_n \ge 0$ for $n \in \mathbb{N}$. We easily check that $a_1 = a_2 = 0$ and

$$a_3 = \frac{16p(5p^2-1)}{(p+1)^4(3p^2+12p+5)} \ge 0. \tag{2.13}$$

Assume that $a_n > 0$ for $n \ge 3$. From (2.12) it is easy to see that

$$w(n,p) \ge w\left(n,\frac{1}{\sqrt{5}}\right) = \frac{4}{5}(n-2) > 0,$$

which together with (2.11) yields $v(n,p) > 0$, and from (2.10) it is derived that $a_{n+1} > 0$. By mathematical induction, we conclude that $a_n \geq 0$ for $n \in \mathbb{N}$, which proves part one of this lemma.

(ii) Next we show that $f$ is decreasing if $p \leq 1/3$. Likewise, it needs to be shown that $a_n \leq 0$ for $n \in \mathbb{N}$. As mentioned previously, $a_1 = a_2 = 0$ but

$$a_3 = \frac{16p(5p^2 - 1)}{(p+1)^4(3p^2 + 12p + 5)} < 0.$$

Suppose that $a_n < 0$ for $n \geq 3$. We have

$$w(n,p) \leq w\left(n, \frac{1}{3}\right) = -\frac{4}{3} < 0,$$

which leads to $v(n,p) < 0$, and from (2.10) we have $a_{n+1} < 0$ for $n \geq 3$. By mathematical induction, it is obtained that $a_n \leq 0$ for $n \in \mathbb{N}$.

This completes the proof. □

Now we prove Theorem 1′.

*Proof of Theorem* 1′ It is clear that (1.16) (or its reverse inequality) is equivalent to $f(t) > 0$ (or $< 0$), where $f(t)$ is defined by (2.4).

(i) We first show that $f(t) > 0$ for all $t > 0$ if and only if $p \geq 1/\sqrt{5}$. If $f(t) > 0$ for all $t > 0$, then by (2.5) and (2.6) we have

$$\begin{cases} \lim_{t\to 0^+} \frac{f(t)}{t^4} = \frac{1}{36}(p^2 - \frac{1}{5}) \geq 0, \\ \lim_{t\to\infty} \frac{f(t)}{t} = \frac{1}{p}(p - \frac{1}{3}) \geq 0, \end{cases}$$

which yields $p \geq 1/\sqrt{5}$.

Conversely, if $p \geq 1/\sqrt{5}$, then by Lemma 4 $f$ is increasing on $(0,\infty)$, hence

$$f(t) > \lim_{t\to 0^+} f(t) = 0$$

for all $t > 0$.

(ii) Next we prove that $f(t) < 0$ for all $t > 0$ if and only if $p \leq 1/3$. If $f(t) < 0$ for all $t > 0$, then by (2.5) and (2.6) we have

$$\begin{cases} \lim_{t\to 0^+} \frac{f(t)}{t^4} = \frac{1}{36}(p^2 - \frac{1}{5}) \leq 0, \\ \lim_{t\to\infty} \frac{f(t)}{t} = \frac{1}{p}(p - \frac{1}{3}) \leq 0, \end{cases}$$

which leads to $0 < p \leq 1/3$.

Conversely, if $0 < p \leq 1/3$, then from the monotonicity of $f$ by Lemma 4 we conclude that

$$f(t) < \lim_{t\to 0^+} f(t) = 0$$

for all $t > 0$.

(iii) Lastly, from Lemma 2 we easily conclude that the function $p \mapsto (\cosh pt)^{1/(3p^2)}$ is decreasing on $(0, \infty)$.

Thus the proof is accomplished. □

## 3 Proof of Theorem 2′

The following lemmas are useful.

**Lemma 5** *Let* $p > 0$ *and let* $g : (0, \infty) \mapsto (-\infty, \infty)$ *be the function defined by*

$$g(t) = \left( \frac{t \cosh t}{\sinh t} - 1 \right) - \frac{2}{3p^2} \ln(\cosh pt). \tag{3.1}$$

*Then we have*

$$\lim_{t \to 0^+} \frac{g(t)}{t^4} = \frac{1}{18} \left( p^2 - \frac{2}{5} \right), \tag{3.2}$$

$$\lim_{t \to \infty} \frac{g(t)}{t} = \frac{1}{p} \left( p - \frac{2}{3} \right). \tag{3.3}$$

*Proof* Since $g(t) \to 0$ as $t \to 0^+$, using L'Hospital's rule yields (3.2). To obtain (3.3), we have to change $g(t)$ as follows:

$$g(t) = \left( 1 - \frac{2}{3p} \right) t + \left( \frac{2t}{e^{2t} - 1} - 1 \right) - \frac{2}{3p^2} \ln \frac{1 + e^{-2pt}}{2}, \tag{3.4}$$

from which (3.3) follows.

Thus the lemma is proved. □

**Lemma 6** *For* $p > 0$, *let the function* $g$ *be defined by* (3.1). *Then* $g$ *is increasing if* $p \geq 2/3$ *and decreasing if* $0 < p \leq \sqrt{10}/5$.

*Proof* Differentiation yields

$$g'(t) = -\frac{g_1(t)}{3p \cosh pt \sinh^2 t}, \tag{3.5}$$

where

$$\begin{aligned} g_1(t) = {} & 2 \sinh pt \sinh^2 t - 3p \cosh pt \cosh t \sinh t \\ & + 3pt \cosh pt \cosh^2 t - 3pt \cosh pt \sinh^2 t. \end{aligned}$$

Using 'product into sum' formula for hyperbolic functions and expanding in power series give

$$\begin{aligned} g_1(t) &= \left( \frac{1}{2} + \frac{3}{4} p \right) \sinh t(p-2) + \left( \frac{1}{2} - \frac{3}{4} p \right) \sinh t(p+2) - \sinh pt + 3pt \cosh pt \\ &= \left( \frac{1}{2} + \frac{3}{4} p \right) \sum_1^\infty \frac{t^{2n-1}(p-2)^{2n-1}}{(2n-1)!} - \left( \frac{3}{4} p - \frac{1}{2} \right) \sum_1^\infty \frac{t^{2n-1}(p+2)^{2n-1}}{(2n-1)!} \end{aligned}$$

$$\begin{aligned}
&-\sum_{1}^{\infty}\frac{t^{2n-1}p^{2n-1}}{(2n-1)!}+3p\sum_{1}^{\infty}\frac{t^{2n-1}p^{2n-2}}{(2n-2)!}\\
:=&\sum_{1}^{\infty}\frac{b_n}{(2n-1)!}p^{2n-1}t^{2n-1},
\end{aligned}\tag{3.6}$$

where

$$b_n=\left(\frac{1}{2}+\frac{3}{4}p\right)\left(1-\frac{2}{p}\right)^{2n-1}-\left(\frac{3}{4}p-\frac{1}{2}\right)\left(1+\frac{2}{p}\right)^{2n-1}+2(3n-2).\tag{3.7}$$

We find that

$$b_{n+1}=\left(1-\frac{2}{p}\right)^{2}b_n+h(n,p),\tag{3.8}$$

where

$$h(n,p)=-\frac{2(3p-2)}{p}\left(1+\frac{2}{p}\right)^{2n-1}+\frac{24(p-1)n+2(3p^2-8p+8)}{p^2}.$$

We claim that

$$h(n,p)<0\quad\text{if } p\ge 2/3,\tag{3.9}$$

$$h(n,p)>0\quad\text{if } 0<p\le\sqrt{10}/5.\tag{3.10}$$

Indeed, applying the binomial expansion gives

$$\left(1+\frac{2}{p}\right)^{2n-1}\ge 1+(2n-1)\frac{2}{p}+\frac{(2n-1)(2n-2)}{2!}\left(\frac{2}{p}\right)^{2}.$$

Hence, if $p\ge 2/3$, then we get

$$\begin{aligned}
h(n,p)&<-\frac{2(3p-2)}{p}\left(1+(2n-1)\frac{2}{p}+\frac{(2n-1)(2n-2)}{2!}\left(\frac{2}{p}\right)^{2}\right)\\
&\quad+\frac{24(p-1)n+2(3p^2-8p+8)}{p^2}\\
&=\frac{16}{p^3}(n-1)\left((3n-1)\left(\frac{2}{3}-p\right)-\frac{1}{3}\right)<0,
\end{aligned}$$

that is, (3.9) holds. If $0<p\le\sqrt{10}/5$, then

$$\begin{aligned}
h(n,p)&>-\frac{2(3p-2)}{p}\left(1+(2n-1)\frac{2}{p}+\frac{(2n-1)(2n-2)}{2!}\left(\frac{2}{p}\right)^{2}\right)\\
&\quad+\frac{24(p-1)n+2(3p^2-8p+8)}{p^2}\\
&=\frac{16}{p^3}(n-1)\left((3n-1)\left(\frac{2}{3}-p\right)-\frac{1}{3}\right)
\end{aligned}$$

$$\begin{aligned}&\geq 16\left(\frac{5}{\sqrt{10}}\right)^3(n-1)\left((3n-1)\left(\frac{2}{3}-\frac{\sqrt{10}}{5}\right)-\frac{1}{3}\right)\\&=40(\sqrt{10}-3)(n-1)\left(n-\frac{4+\sqrt{10}}{2}\right)>0\end{aligned}$$

for $n \geq 4$, which in combination with

$$\begin{aligned}h(1,p)&=-\frac{2}{p^2}(3p-2)(p+2)>0,\\h(2,p)&=-\frac{2}{p^3}(3p-2)\left(p^2+6p+12\right)>0,\\h(3,p)&=-\frac{2}{p^3}(3p-2)\left(p^2+10p+40\right)>0\end{aligned}$$

leads to (3.10).

Now we are in a position to prove our results.

(i) We first prove that $g$ is increasing if $p \geq 2/3$. For this end, it is enough to show that $b_n \leq 0$ by (3.5) and (3.6). Indeed, we have $b_1 = b_2 = 0$ and $b_3 = -16p^{-4}(5p^2-2) < 0$. Suppose that $b_n \leq 0$ for $n \geq 4$. From (3.8) and (3.9) we have $b_{n+1} < 0$, which proves part one of this lemma by mathematical induction.

(ii) Next we prove that $g$ is decreasing if $0 < p \leq \sqrt{10}/5$. It suffices to prove that $b_n \geq 0$. We have seen that $b_1 = b_2 = 0$, but $b_3 = -16p^{-4}(5p^2-2) \geq 0$. Using (3.8) and (3.10), we conclude that $b_{n+1} \geq 0$ if $b_n \geq 0$ for $n \geq 3$. By mathematical induction, part two of this lemma is proved.

Thus the proof ends. □

Based on the above lemmas, Theorem 2′ can be easily proved.

*Proof of Theorem* 2′ It is clear that (1.17) (or its reverse inequality) is equivalent to $g(t) > 0$ (or $< 0$), where $g(t)$ is defined by (3.1).

(i) We first prove that $g(t) > 0$ for all $t > 0$ if and only if $p \geq 2/3$. If $g(t) > 0$ for all $t > 0$, then by (3.2) and (3.3) we have

$$\begin{cases}\lim_{t\to 0^+}\frac{g(t)}{t^4}=\frac{1}{18}(p^2-\frac{2}{5})\geq 0,\\ \lim_{t\to\infty}\frac{g(t)}{t}=\frac{1}{p}(p-\frac{2}{3})\geq 0,\end{cases}$$

which leads to $p \geq 2/3$.

Conversely, if $p \geq 2/3$, then by Lemma 6 we get

$$g(t) > \lim_{t\to 0^+} g(t) = 0$$

for all $t > 0$.

(ii) Next we show that $g(t) < 0$ for all $t > 0$ if and only if $0 < p \leq \sqrt{10}/5$. If $g(t) < 0$ for all $t > 0$, then by (3.2) and (3.3) we obtain

$$\begin{cases}\lim_{t\to 0^+}\frac{g(t)}{t^4}=\frac{1}{18}(p^2-\frac{2}{5})\leq 0,\\ \lim_{t\to\infty}\frac{g(t)}{t}=\frac{1}{p}(p-\frac{2}{3})\leq 0.\end{cases}$$

Solving the inequalities yields $0 < p \leq \sqrt{10}/5$.

Conversely, if $0 < p \le \sqrt{10}/5$, then by the monotonicity of $g$ we have

$$g(t) < \lim_{t\to 0^+} g(t) = 0$$

for all $t > 0$.

(iii) Lastly, due to Lemma 2, the function $p \mapsto (\cosh pt)^{1/(3p^2)}$ is clearly decreasing on $(0,\infty)$.

This proves the proof. □

## 4 Corollaries

Using Theorem 1 and (2.2), the following corollaries are immediate.

**Corollary 1** *We have*

$$\begin{aligned} G &< \cdots < A^{1/3}G^{2/3} < A_{1/2}^{2/3}G^{1/3} < A_{7/15}^{5/7}G^{2/7} < A_{p_0}^{1/(3p_0)}G^{1-1/(3p_0)} < L < A_{1/3} \\ &< \cdots < G\exp\left(\frac{1}{24}\ln^2(x/y)\right) \end{aligned} \tag{4.1}$$

*holds for* $x,y > 0$ *with* $x \ne y$, *where the constants* $p_0 = 1/\sqrt{5}$ *and* $1/3$ *are the best constants.*

By Theorem 2 and (2.3), we obtain the following.

**Corollary 2** *We have*

$$\begin{aligned} G &< \cdots < A^{2/3}G^{1/3} < A_{2/3} < I < A_{q_0}^{2/(3q_0)}G^{1-2/(3q_0)} < A_{1/2}^{4/3}G^{-1/3} \\ &< \cdots < G\exp\left(\frac{1}{12}\ln^2(x/y)\right) \end{aligned} \tag{4.2}$$

*holds for* $x,y > 0$ *with* $x \ne y$, *where* $2/3$ *and* $q_0 = \sqrt{10}/5$ *are the best constants.*

**Remark 2** Neuman [9] has derived some bounds for certain differences of bivariate means, one of which is as follows:

$$\frac{\min(x,y)}{24}\ln^2(x/y) < L - G < \frac{\max(x,y)}{24}\ln^2(x/y). \tag{4.3}$$

While (4.1) and (4.2) contain some new bounds for certain ratios of bivariate means, for example,

$$1 < \frac{L}{G} < \exp\left(\frac{1}{24}\ln^2(x/y)\right), \tag{4.4}$$

$$1 < \frac{I}{G} < \exp\left(\frac{1}{12}\ln^2(x/y)\right), \tag{4.5}$$

where $x,y > 0$ with $x \ne y$.

Making use of identity for means

$$\ln\frac{I}{G} = \frac{A}{L} - 1$$

given in [34, 35], inequality (4.5) can be changed as follows:

$$1 < \frac{A}{L} < 1 + \frac{1}{12}\ln^2(x/y). \tag{4.6}$$

Employing Theorem 1, Theorem 2 and (2.2), we can prove an interesting chain of inequalities involving the logarithmic mean, identric mean, power mean and geometric mean.

**Corollary 3** *Let $p \geq 2/3$, $1/\sqrt{5} \leq q \leq \sqrt{10}/5$, $0 < r \leq \sqrt{10}/10$. Then the inequalities*

$$\begin{aligned} G &< \cdots < A_p^{1/(3p)} G^{1-1/(3p)} < \sqrt{IG} < A_q^{1/(3q)} G^{1-1/(3q)} < L < A_{1/3} < I_{1/2} < A_r^{1/(3r)} G^{1-1/(3r)} \\ &< \cdots < G\exp\left(\frac{1}{24}\ln^2(x/y)\right) \end{aligned} \tag{4.7}$$

*hold for $x, y > 0$ with $x \neq y$, where $I_{1/2} = I^2(\sqrt{x}, \sqrt{y})$.*

*Proof* By Remark 1, we see that the function $p \mapsto A_p^{1/(3p)} G^{1-1/(3p)}$ is decreasing on $(0, \infty)$, and by (2.2) it is deduced that

$$G < \cdots < A_p^{1/(3p)} G^{1-1/(3p)}, \qquad A_r^{1/(3r)} G^{1-1/(3r)} < \cdots < G\exp\left(\frac{1}{24}\ln^2(x/y)\right)$$

if $p \geq 2/3$ and $0 < r \leq \sqrt{10}/10$.

The second and third inequalities are equivalent to (1.17), which hold if and only if $p \geq 2/3$ and $q \leq \sqrt{10}/5$ by Theorem 2, respectively.

If $1/\sqrt{5} \leq q \leq \sqrt{10}/5$, then by Theorem 1 the fourth and fifth inequalities hold.

With $x \to x^{1/2}$, $y \to y^{1/2}$, $r \to 2r$, by Theorem 2, we have

$$A^{3/2}\left(\sqrt[3]{x}, \sqrt[3]{y}\right) < I(\sqrt{x}, \sqrt{y}) \tag{4.8}$$

and for $0 < 2r \leq \sqrt{10}/5$, that is, $0 < r \leq \sqrt{10}/10$,

$$\begin{aligned} I(\sqrt{x}, \sqrt{y}) &< \left(A^{1/(2r)}\left(x^{1/r}, y^{1/r}\right)\right)^{2/(3(2r))} G^{1-2/(3(2r))}(\sqrt{x}, \sqrt{y}) \\ &= \left(A_r^{1/(3r)} G^{1-1/(3r)}\right)^{1/2}. \end{aligned} \tag{4.9}$$

Squaring both sides of (4.8) and (4.9) yields the sixth and seventh inequality, respectively.

The proof is finished. □

From Lemma 6 with (3.4) another known interesting inequality can be reobtained. It should be noted that the second inequality in (4.10) first appeared in [33] and was reproved by Neuman and Sándor [36].

**Corollary 4** *For $x, y > 0$ with $x \neq y$, we have*

$$A_{2/3} < I < 2\sqrt{2}e^{-1} A_{2/3}, \tag{4.10}$$

*where $2\sqrt{2}e^{-1}$ is the best constant.*

*Proof* From (3.4) it is obtained that

$$\lim_{t\to\infty} g(t) = \frac{3}{2}\ln 2 - 1$$

if $p = 2/3$. And since the function $t \mapsto g(t)$ is increasing on $(0,\infty)$ by Lemma 6, we have

$$0 = \lim_{t\to\infty} g(t) < g(t) < \lim_{t\to\infty} g(t) = \frac{3}{2}\ln 2 - 1,$$

that is,

$$\left(\cosh\frac{2}{3}t\right)^{3/2} < e^{\frac{t\cosh t}{\sinh t}-1} < 2\sqrt{2}e^{-1}\left(\cosh\frac{2}{3}t\right)^{3/2}, \tag{4.11}$$

which is equivalent to (4.10).

Thus the proof is completed. □

## 5 Comparison of some lower bounds for logarithmic mean

As mentioned in the first section of this paper, there are many lower bounds for the logarithmic mean $L$ such as

$$G, \quad \sqrt{A_{1/2}G}, \quad A^{1/3}G^{2/3}, \quad A_p^{1-p}G^p, \quad \left((7A+8G)/15\right)^{5/7}G^{2/7},$$
$$A_{1/2}^{2/3}G^{1/3}, \quad \sqrt{IG}, \quad \sqrt{\frac{2A+G}{3}G},$$

and so on, some of which have been proved to be comparable and others remain to be compared further. As applications of our main results, we will discuss them in this section. To this end, we first give a lemma.

**Lemma 7** ([33, Conclusion 1]) *The function $r \mapsto A_r$ is strictly log-concave on $[0,\infty)$.*

Now we compare $A_{p_0}^{1/(3p_0)}G^{1-1/(3p_0)}$ with $((7A+8G)/15)^{5/7}G^{2/7}$.

**Lemma 8** *Let $p > 0$. Then the inequalities*

$$A_{p_0}^{1/(3p_0)}G^{1-1/(3p_0)} > \left(\frac{7A+8G}{15}\right)^{5/7}G^{2/7} > A_{p_1}^{1/(3p_1)}G^{1-1/(3p_1)} \tag{5.1}$$

*hold for all $x, y > 0$ with $x \neq y$, where $p_0 = 1/\sqrt{5}$ and $p_1 = 7/15$ cannot be improved.*

*Proof* By Lemma 1, to prove (5.1), it suffices to show that

$$\frac{1}{3p_0^2}\ln(\cosh p_0 t) > \frac{5}{7}\ln\frac{7\cosh t + 8}{15} > \frac{1}{3p_1^2}\ln(\cosh p_1 t). \tag{5.2}$$

For $t > 0$, we define

$$D_1(t) := \frac{1}{3p^2}\ln(\cosh pt) - \frac{5}{7}\ln\frac{7\cosh t + 8}{15}.$$

Differentiation and expanding in power series lead to

$$\begin{aligned}
D_1'(t) &= \frac{(8\sinh tp + 7\sinh tp\cosh t - 15p\cosh tp\sinh t)}{3p(\cosh tp)(7\cosh t + 8)} \\
&= \frac{(7-15p)\sinh t(p+1) + 16\sinh tp - (7+15p)\sinh t(1-p)}{6p(\cosh tp)(7\cosh t + 8)} \\
&= \frac{1}{6p(\cosh tp)(7\cosh t + 8)} \sum_1^{\infty} \frac{p^{2n-1} v_n}{(2n-1)!} t^{2n-1},
\end{aligned}$$

where

$$v_n = (7-15p)\left(1+\frac{1}{p}\right)^{2n-1} - (7+15p)\left(\frac{1}{p}-1\right)^{2n-1} + 16.$$

We easily establish a recursive relation for the sequence $(v_n)$:

$$v_{n+1} = v_n\left(\frac{1}{p}-1\right)^2 + w_n, \tag{5.3}$$

where

$$w_n = \frac{4}{p}(7-15p)\left(1+\frac{1}{p}\right)^{2n-1} - \frac{16}{p}(1-2p).$$

Clearly, if we prove that for all $n \in \mathbb{N}$, $v_n \geq 0$ if $p = p_0 = 1/\sqrt{5}$ and $v_n \leq 0$ if $p = p_1 = 7/15$, then inequalities (5.1) are valid.

Now we show that for all $n \in \mathbb{N}$, $v_n \geq 0$ if $p = p_0 = 1/\sqrt{5}$. In fact, it is easy to verify that $v_1 = v_2 = 0$, $v_3 = 80 > 0$ and due to $(7 - 15p_0) > 0$,

$$w_n \geq w_2 = \frac{4}{p_0}(7-15p_0)\left(1+\frac{1}{p_0}\right)^3 - \frac{16}{p_0^2}(1-2p_0) = 80 > 0,$$

which together with (5.3) yields $v_{n+1} > v_n(\frac{1}{p_0}-1)^2 > 0$ under the inductive assumption $v_n > 0$ for $n \geq 3$. By mathematical induction, it is acquired that $v_n \geq 0$ for all $n \in \mathbb{N}$.

We next prove that for all $n \in \mathbb{N}$, $v_n \leq 0$ if $p = p_1 = 7/15$. It is not difficult to get

$$v_n = 16 - 14\left(\frac{8}{7}\right)^{2n-1} = 16\left(1-\left(\frac{8}{7}\right)^{2n-2}\right) \leq 0$$

for all $n \in \mathbb{N}$.

Lastly, we prove $p_0 = 1/\sqrt{5}$ and $p_1 = 7/15$ cannot be improved. Indeed, if $D_1(t) > 0$ for all $t > 0$, then we have

$$\begin{aligned}
\lim_{t\to 0^+} \frac{D_1(t)}{t^4} &= -\frac{1}{180}\left(5p^2 - 1\right) \geq 0, \\
\lim_{t\to 0^+} \frac{D_1(t)}{t^4} &= -\frac{1}{21p}(15p-7) \geq 0,
\end{aligned}$$

which yields $0 < p \le p_0 = 1/\sqrt{5}$. On the other hand, by Lemma 2 the function $p \mapsto \frac{1}{3p^2}\ln(\cosh pt)$ is decreasing on $(0,\infty)$. Therefore, $p_0 = 1/\sqrt{5}$ is the largest constant such that $D_1(t) > 0$ for all $t > 0$.

In the same way, we can prove $p_1 = 7/15$ is the smallest constant such that $D_1(t) < 0$ for all $t > 0$.

This completes the proof. □

Next let us compare $\sqrt{\frac{2A+G}{3}G}$ and $A_{2/3}^{2/3}A_{1/3}^{-1/3}G^{2/3}$.

**Lemma 9** *For $x, y > 0$ with $x \neq y$, we have*

$$\sqrt{\frac{2A+G}{3}G} > A_{2/3}^{2/3}A_{1/3}^{-1/3}G^{2/3}. \tag{5.4}$$

*Proof* Suppose that $x > y > 0$ and let $x/y = t^6$. Then inequality (5.4) can be equivalently changed into

$$D_2(t) = \frac{t^6+t^3+1}{3} - t\left(\frac{t^4+1}{t^2+1}\right)^2 > 0,$$

where $t > 1$. Simplifying yields

$$D_2(t) = \frac{1}{3(t^2+1)^2}(t-1)^4\left(t^6+t^5-t^3+t+1\right) > 0,$$

which completes the proof. □

Using Lemmas 7-9, we can easily prove the following.

**Proposition 1** *For $x, y > 0$ with $x \neq y$ and $p_0 = \sqrt{5}/5$, $p_2 = \sqrt{10}/5 = 0.63246$, we have*

$$\begin{aligned} L &> A_{p_0}^{1/(3p_0)}G^{1-1/(3p_0)} > \left(\frac{7A+8G}{15}\right)^{5/7}G^{2/7} > A_{7/15}^{5/7}G^{2/7} \\ &> A_{1/2}^{2/3}G^{1/3} > A_{p_2}^{1/(3p_2)}G^{1-1/(3p_2)} > \sqrt{IG} > \sqrt{A_{2/3}G} \\ &> \sqrt{\frac{2A+G}{3}G} > A_{2/3}^{2/3}A_{1/3}^{-1/3}G^{2/3} > A^{1/3}G^{2/3}. \end{aligned} \tag{5.5}$$

*Proof* The first, second and third inequalities follow from Theorem 1 and Lemma 8. Since $7/15 < 1/2 < p_2 = \sqrt{10}/5$, by Theorem 1 the fourth and fifth ones follow. The sixth and seventh ones are obtained from the second and third ones of (4.7).

By the known inequality $A_{2/3} > (2A+G)/3$ (see [33, (5.13)]), we easily get the eighth one.

Lemma 9 shows that the ninth one holds.

The last one is equivalent to $A_{2/3} > A^{1/2}A_{1/3}^{1/2}$, which easily follows from Lemma 7.

This completes the proof. □

Lastly, we compare $A_{p_0}^{1/(3p_0)}G^{1-1/(3p_0)}$ with $A_p^{1-p}G^p$ $(0 < p < 1)$.

**Proposition 2** *Let $p_0 = 1/\sqrt{5}$ and $0 < p < 1$. Then*

$$A_{p_0}^{1/(3p_0)} G^{1-1/(3p_0)} > A_p^{1-p} G^p \tag{5.6}$$

*holds for all $x, y > 0$ with $x \neq y$ if $1 - 1/(3p_0) \leq p < 1$. While $A_{p_0}^{1/(3p_0)} G^{1-1/(3p_0)}$ and $A_p^{1-p} G^p$ are not comparable if $0 < p < 1 - 1/(3p_0)$.*

*Proof* (i) By a simple equivalent transformation, inequality (5.6) can be changed into

$$A_{p_0} > A_p^{3p_0(1-p)} G^{3p_0p-3p_0+1} := A_p^\alpha A_0^\beta,$$

where $\alpha = 3p_0(1-p)$, $\beta = 3p_0p - 3p_0 + 1$. If $1 - 1/(3p_0) \leq p < 1$, then $\alpha \geq 0$, $\beta > 0$ with $\alpha + \beta = 1$. By Lemma 7, it is derived that

$$A_p^\alpha A_0^\beta < A_{\alpha p} = A_{3p_0(1-p)p},$$

and using the basic inequality $(1-p)p \leq 1/4$ for $0 < p < 1$ and the monotonicity of the function $r \mapsto A_r$, we get

$$A_{3p_0(1-p)p} \leq A_{3p_0/4} < A_{p_0},$$

which proves inequality (5.6).

(ii) We define

$$D_3(t) := \frac{1}{3p_0^2} \ln(\cosh p_0 t) - \frac{1-p}{p} \ln\cosh pt.$$

By Lemma 1, to show that $A_{p_0}^{1/(3p_0)} G^{1-1/(3p_0)}$ and $A_p^{1-p} G^p$ are not comparable if $0 < p < 1 - 1/(3p_0)$, we need to illustrate $\operatorname{sgn}(D_3(t))$ is not a constant. In fact, utilizing L'Hospital's rule, we have

$$\lim_{t\to 0^+} \frac{D_3(t)}{t^2} = \frac{1}{2}(p - 1/2)^2 + \frac{1}{24} > 0,$$
$$\lim_{t\to\infty} \frac{D_3(t)}{t} = \frac{1}{3p_0} + (p-1) < 0,$$

which implies that there are numbers $t_2 > t_1 > 0$ such that $D_3(t) > 0$ when $t \in (0, t_1)$ and $D_3(t) < 0$ when $t \in (t_2, \infty)$. Consequently, $\frac{1}{3p_0^2} \ln(\cosh p_0 t)$ and $\frac{1-p}{p} \ln\cosh pt$ are not comparable on $(0, \infty)$ if $0 < p < 1 - 1/(3p_0)$, which is the desired result.

Thus the proof is finished. □

**Remark 3** From the above two propositions, as far as the lower bounds for the logarithmic mean are concerned, our new lower bound $A_{p_0}^{1/(3p_0)} G^{1-1/(3p_0)}$ seems to be superior to most known ones.

**Competing interests**

The author declares that they have no competing interests.

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